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PENNY-SHAPED CRACK IN A LINEAR VISCOELASTIC MEDIUM UNDER SHEAR

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Abstract—The problem of a linear viscoelastic body, containing a penny-shaped crack subjected to the shear parallel to the edge of the crack is considered in this paper. Closed form expressions for the displacements over the surface of the crack, the shear components in the plane of the crack and the stress intensity factors are determined. The various expressions are then specialized for two particular linear viscoelastic materials and the effect of viscoelasticity, wherever possible, is pointed out.

1. INTRODUCTION

The classical method of solving viscoelastic stress analysis problems by the use of integral transforms, generally referred to as the "correspondence principle" in the literature, is no longer applicable in solving mixed boundary-value problems where the boundaries vary with time. Graham[1] has proposed a correspondence principle of linear viscoelasticity theory for mixed boundary-value problems involving time dependent boundary regions and has applied it[2] to solve the problems of penny-shaped crack subjected to a tension normal to the plane of the crack. Graham's principle is, however, applicable to a restricted class of deformations since it requires a very special form of the stress and the displacement distribution for the elastic solution. In view of the fact that in the case of the problem of a penny-shaped crack under uniform shear the elastic solutions, as presented by Segedin[3] and Westmann[4], do not possess the special features, Graham's correspondence principle cannot be applied in this case.

Ting[5] has developed a technique of solving the problems with moving boundaries and has applied it to obtain the contact stresses between an axysymmetric rigid identer and a viscoelastic half-space. Ting's method is to replace the time dependent boundary condition by an alternate boundary condition for which the integral transform technique is applicable. The problem is then solved in terms of this unknown boundary condition and the determination of it reduces in finding the solutions of integral equations. Ting[6] used the method to solve the problem of a viscoelastic hollow cylinder with ablating inner radius and the present authors[7] have extended the technique to solve the problem of a penny-shaped crack in a viscoelastic medium under torsion.

In the present paper we apply a method, similar in the spirit to that of Ting[5], to solve the problem of a penny-shaped crack in a linear viscoelastic medium under uniform shear. Expressions for the displacements over the surface of the crack, the stress components in the plane of the crack and the stress intensity factors are given for the general linear viscoelastic medium. The various expressions are then specialized for a Maxwell material and a standard linear solid and the effect of viscoelasticity is also briefly pointed out.

2. BASIC EQUATIONS

Suppose a region R with boundary B is occupied by a homogeneous and isotropic linear viscoelastic solid. Let u_i, e_{ij}, σ_{ij}, each of which is a function of the position vector $\mathbf{x}$ and time t where $\mathbf{x}$ is a point in R and $0 \leqslant t < \infty$, denote the components of displacement, strain and stress respectively. Then the relevant field equations and the constitutive equations appropriate to the linear quasistatic theory of viscoelasticity, in the absence of body forces, are:

$$2e_{ij}(\mathbf{x}, t) = u_{i,j}(\mathbf{x}, t) + u_{j,i}(\mathbf{x}, t), \tag{1}$$

$$\sigma_{ij,j}(\mathbf{x}, t) = 0, \qquad \sigma_{ij}(\mathbf{x}, t) = \sigma_{ji}(\mathbf{x}, t), \tag{2}$$

$$\sigma_{ij}(\mathbf{x}, t) = G_1 * \mathrm{d}e_{ij}(\mathbf{x}, t) + \tfrac{1}{3}\delta_{ij}(G_2 - G_1) * \mathrm{d}e_{kk}(\mathbf{x}, t). \tag{3}$$

Here $G_1(t)$ and $G_2(t)$ are the relaxation functions in shear and isotropic compressions respectively. Also the Stieltjes convolution, $g * \mathrm{d}h$, of two functions $g(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$, is defined by

$$[g * \mathrm{d}h](\mathbf{x}, t) = g(\mathbf{x}, t)h(\mathbf{x}, 0) + \int_0^t g(\mathbf{x}, t-\tau)\frac{\partial h}{\partial \tau}(\mathbf{x}, \tau)\,\mathrm{d}\tau. \tag{4}$$

We use the notation

$$\bar{g}(\mathbf{x}, p) = L\{g(\mathbf{x}, t); t \to p\} = \int_0^\infty g(\mathbf{x}, t)\,\mathrm{e}^{-pt}\,\mathrm{d}t, \tag{5}$$

for the Laplace transform with respect to time of the function $g(\mathbf{x}, t)$. On taking the Laplace transform of equations (1–3), we have

$$2\bar{e}_{ij}(\mathbf{x}, p) = \bar{u}_{i,j}(\mathbf{x}, p) + \bar{u}_{j,i}(\mathbf{x}, p), \tag{6}$$

$$\bar{\sigma}_{ij,j}(\mathbf{x}, p) = 0, \qquad \bar{\sigma}_{ij}(\mathbf{x}, p) = \bar{\sigma}_{ji}(\mathbf{x}, p), \tag{7}$$

$$\bar{\sigma}_{ij}(\mathbf{x}, p) = p\bar{G}_1(p)\bar{e}_{ij}(\mathbf{x}, p) + \tfrac{1}{3}\delta_{ij}p\{\overline{G_2(p) - G_1(p)}\}\bar{e}_{kk}(\mathbf{x}, p). \tag{8}$$

From equation (8) we note that if we set

$$3\bar{\lambda} = p\{\overline{G_2(p) - G_1(p)}\}, \qquad 2\bar{\mu} = p\bar{G}_1(p), \tag{9}$$

then the equations (6–9) have similar structure as to the equations governing the classical linear theory of elasticity.

3. PENNY-SHAPED CRACK UNDER SHEAR

Consider an infinite linear viscoelastic medium containing a penny-shaped crack which is subjected to a shearing stress $\sigma_{xz} \sim S(t)$ as $\sqrt{(x^2 + y^2 + z^2)} \to \infty$. We choose cylindrical polar coordinates (ρ, θ, z) such that the crack occupies the region $0 \leqslant \rho \leqslant a(t)$, $z = 0$, for all θ. It is clear that the solution of this problem is equivalent to finding the stress distribution and the displacement components in a half space in which the shearing stress is acting on the crack surfaces, all the displacement components exterior to the crack surface are zero, and the entire crack surface is free from the normal tractions. In terms of the coordinate system chosen, these boundary conditions can be expressed as:

$$\left.\begin{aligned}\sigma_{\rho z} &= S(t)\cos\theta \\ \sigma_{\theta z} &= -S(t)\sin\theta\end{aligned}\right\} z = 0, \qquad 0 \leqslant \rho \leqslant a(t). \tag{10}$$

$$\sigma_{zz} = 0, \qquad z = 0, \qquad \forall \rho. \tag{11}$$

$$\left.\begin{aligned} u_\rho &= 0 \\ u_\theta &= 0 \end{aligned}\right\} z = 0, \qquad \rho > a(t). \tag{12}$$

Here u_ρ and u_θ are the radial and the angular displacement components respectively and $S(t)$ is known quantity. We also assume that the radius of the crack $a(t)$ is a monotonic increasing function of time.

In order to solve equations (6–9) we assume the displacement components to be as:

$$2\bar{\mu}\bar{u}_\rho(\rho, \theta, z, p) = \phi(\rho, z, p) \cos\theta + z \frac{\partial \phi_2}{\partial \rho}(\rho, z, p) \cos\theta, \tag{13}$$

$$2\bar{\mu}\bar{u}_\theta(\rho, \theta, z, p) = -\bar{\phi}(\rho, z, p) \sin\theta + (z/\rho)\, \phi_2(\rho, z, p) \sin\theta, \tag{14}$$

$$2\bar{\mu}\bar{u}_z(\rho, \theta, z, p) = \phi_1(\rho, z, p) \cos\theta + z \frac{\partial \phi_2}{\partial z}(\rho, z, p) \cos\theta, \tag{15}$$

where the scalar functions $\phi(\rho, z, p)$, $\phi_1(\rho, z, p)$, and $\phi_2(\rho, z, p)$ satisfy Laplace equation and are connected by the relation

$$\frac{\partial \phi}{\partial \rho} + \frac{\partial \phi_1}{\partial z} + (3 - 4\bar{\nu}) \frac{\partial \phi_2}{\partial z} = 0. \tag{16}$$

Here $\bar{\nu} = \bar{\lambda}/2(\bar{\lambda} + \bar{\mu})$ and $\bar{\lambda}$ and $\bar{\mu}$ are defined by (9). Since σ_{zz} and hence its Laplace transform $\bar{\sigma}_{zz} = 0$ for all ρ, on $z = 0$, it follows from (8), (11), (13–15) that this requirement is equivalent to

$$\left(\frac{\bar{\nu}}{1 - \bar{\nu}}\right) \frac{\partial \phi}{\partial \rho} + \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z} = 0. \tag{17}$$

From (16) and (17) we have

$$\frac{\partial \phi}{\partial \rho} + 2(1 - \bar{\nu}) \frac{\partial \phi_2}{\partial z} = 0. \tag{18}$$

Equation (18), at once, suggests that we can choose

$$\phi = -2(1 - \bar{\nu}) \frac{\partial \bar{\Phi}}{\partial z}, \tag{19}$$

$$\phi_2 = \frac{\partial \bar{\Phi}}{\partial \rho},$$

where $\bar{\Phi}(\rho, z, p)$ satisfies the equation:

$$\frac{\partial^2 \bar{\Phi}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \bar{\Phi}}{\partial \rho} + \frac{\partial^2 \bar{\Phi}}{\partial z^2} = 0. \tag{20}$$

Also on employing (19) into (16) we get

$$\phi_1 = -(1 - 2\bar{\nu}) \frac{\partial \bar{\Phi}}{\partial \rho}. \tag{21}$$

Thus the displacement components can now be written in terms of a single scalar function $\bar{\Phi}(\rho, z, p)$, as:

$$2\bar{\mu}\bar{u}_\rho(\rho, \theta, z, p) = -2(1-\bar{\nu})\frac{\partial\bar{\Phi}}{\partial z}\cos\theta + z\frac{\partial^2\bar{\Phi}}{\partial\rho^2}\cos\theta,$$

$$2\bar{\mu}\bar{u}_\theta(\rho, \theta, z, p) = 2(1-\bar{\nu})\frac{\partial\bar{\Phi}}{\partial z}\sin\theta + \frac{z}{\rho}\frac{\partial\bar{\Phi}}{\partial\rho}\sin\theta, \tag{22}$$

$$2\bar{\mu}\bar{u}_z(\rho, \theta, z, p) = -(1-2\bar{\nu})\frac{\partial\bar{\Phi}}{\partial\rho}\cos\theta + z\frac{\partial^2\bar{\Phi}}{\partial z\partial\rho}\cos\theta.$$

On taking the Hankel transform of order zero of equation (20) and solving the resulting equation, we obtain

$$\hat{\bar{\Phi}}(\xi, z, p) = A\,\mathrm{e}^{-\xi z}, \tag{23}$$

where

$$\hat{\bar{\Phi}}(\xi, z, p) = H_0\{\bar{\Phi}(\rho, z, p); \rho \to \xi\} = \int_0^\infty \rho\bar{\Phi}(\rho, z, p)J_0(\rho\xi)\,\mathrm{d}\rho, \tag{24}$$

and where we have used the fact that displacements should vanish at infinity.

We now consider equation (12) and introduce a unknown function $v_1(\rho, \theta, z, t)$, such that

$$u_\theta(\rho, \theta, z, t) = \begin{cases} 0 & \rho > a(t) \\ v_1(\rho, \theta, z, t), & \rho \leqslant a(t) \end{cases} \quad \text{on } z = 0. \tag{25}$$

Equation (25) can also be written as

$$\bar{u}_\theta(\rho, \theta, z, p) = \begin{cases} 0 & \rho > a(t) \\ \bar{v}_1(\rho, \theta, z, p) & \rho \leqslant a(t) \end{cases} \quad z = 0. \tag{26}$$

A close look between (26) and (22), suggests that it is more appropriate to write

$$\bar{v}_1(\rho, \theta, z, p) = \bar{v}(\rho, z, p)\sin\theta. \tag{27}$$

Hence on equating (26) with $(22)_2$ with the use of (27) and taking the Hankel transform of zero order we get

$$2(1-\bar{\nu})\frac{\partial\hat{\bar{\Phi}}}{\partial z}(\xi, z, p) = 2\bar{\mu}\hat{\bar{v}}(\xi, z, p), \qquad z = 0, \tag{28}$$

where

$$\hat{\bar{v}}(\xi, 0, p) = \int_0^{a(t)} \lambda\bar{v}(\lambda, 0, p)J_0(\xi\lambda)\,\mathrm{d}\lambda. \tag{29}$$

From (24) and (28) we finally have

$$\hat{\bar{\Phi}} = -\frac{\bar{\mu}\hat{\bar{v}}}{(1-\bar{\nu})\xi}\mathrm{e}^{-\xi z}. \tag{30}$$

The stress component $\bar{\sigma}_{\theta z}(\rho, \theta, z, p)$, on employing (22) in (6), (8) and (9), on $z = 0$, is given by

$$\bar{\sigma}_{\theta z}(\rho, \theta, z, p) = \left\{(1-\bar{\nu})\frac{\partial^2\bar{\Phi}}{\partial z^2} - \frac{\bar{\nu}}{\rho}\frac{\partial\bar{\Phi}}{\partial\rho}\right\}\sin\theta. \tag{31}$$

Also on taking the inverse Hankel transform of (30) we get

$$\bar{\Phi} = \frac{-\bar{\mu}}{(1-\bar{\nu})}\int_0^\infty \hat{\bar{v}}(\xi, z, p)\,\mathrm{e}^{-\xi z}J_0(\xi\rho)\,\mathrm{d}\xi. \tag{32}$$

On using (32) in (31) we have

$$\bar{\sigma}_{\theta z}(\rho,\theta,0,p)=\left[-\bar{\mu}\int_0^\infty \xi^2 J_0(\xi\rho)\hat{\bar{v}}(\xi,0,p)\,\mathrm{d}\xi-\frac{\bar{\mu}\bar{v}}{(1-\bar{v})\rho}\int_0^\infty \xi\hat{\bar{v}}(\xi,0,p)J_1(\xi\rho)\,\mathrm{d}\xi\right]\sin\theta. \quad (33)$$

On taking the inverse Laplace transform of (33) and employing the condition that for $\rho\leqslant a(t)$, $\sigma_{\theta z}=-S(t)\sin\theta$, we obtain

$$2S(t)=\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_0(\xi\rho)\int_0^{a(\tau)}\lambda v(\lambda,0,\tau)J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$+\int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \frac{\xi J_1(\xi\rho)}{\rho}\,\mathrm{d}\xi\int_0^{a(\tau)}\lambda v(\lambda,0,\tau)J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau,\qquad \rho\leqslant a(t), \quad (34)$$

where

$$p\frac{\bar{\psi}}{2}=\frac{\bar{\mu}\bar{v}}{(1-\bar{v})}. \quad (35)$$

On writing equation (34) in the form

$$2S(t)=F_1(\rho,t)+F_2(\rho,t), \quad (36)$$

where $F_1(\rho,t)$ and $F_2(\rho,t)$ represent the two right hand side expressions of (34) respectively and setting

$$\begin{aligned}2S(t)-F_1(\rho,t)&=2S_1(\rho,t),\\ 2S(t)-F_2(\rho,t)&=2S_2(\rho,t),\end{aligned} \quad (37)$$

it follows that

$$2S_1(\rho,t)=\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_1(\xi\rho)\int_0^{a(t)}\lambda v(\lambda,0,\tau)J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\xi\,\mathrm{d}\tau,\qquad \rho\leqslant a(t). \quad (38)$$

$$2S_2(\rho,t)=\int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \frac{\xi J_1(\xi\rho)}{\rho}\int_0^{a(\tau)}\lambda v(\lambda,0,\tau)J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\xi\,\mathrm{d}\tau,\qquad \rho\leqslant a(t). \quad (39)$$

We shall now solve the two equations separately.

On changing the variable ρ to m in (38) and multiplying both sides by $[m/(\rho^2-m^2)^{1/2}]$ and integrating as follows, we get

$$\int_0^\rho \frac{2mS_1(m,t)}{\sqrt{\rho^2-m^2}}\,\mathrm{d}m=\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2\,\mathrm{d}\xi\int_0^\rho \frac{mJ_0(\xi m)}{\sqrt{\rho^2-m^2}}\,\mathrm{d}m\int_0^{a(\tau)}\lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\qquad \rho\leqslant a(t). \quad (40)$$

On employing a result from Watson[8] and interchanging the order of integration we have

$$2\int_0^\rho \frac{mS_1(m,t)}{\sqrt{\rho^2-m^2}}\,\mathrm{d}m=-\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\left\{\frac{\partial}{\partial\rho}\int_0^{a(\tau)}\frac{\lambda v(\lambda,0,\tau)}{\sqrt{\lambda^2-\rho^2}}\,\mathrm{d}\lambda\right\}\mathrm{d}\tau. \quad (41)$$

Now again if we change the variable ρ to η and multiply both sides of (41) by $(\eta^2-\rho^2)^{1/2}$ and integrate as follows we get

$$2\int_\rho^{a(t)}\sqrt{\eta^2-\rho^2}\,\mathrm{d}\eta\int_0^\eta \frac{mS_1(m,t)}{\sqrt{\eta^2-m^2}}\,\mathrm{d}m=\frac{\pi}{2}\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(\tau)}\lambda v(\lambda,0,\tau)\,\mathrm{d}\lambda\,\mathrm{d}\tau. \quad (42)$$

Now since $G_1(0) \neq 0$, it follows that there exists a function $G_1^{-1}(t)$, for which

$$G_1(t) * \mathrm{d}G_1^{-1}(t) = G_1^{-1}(t) * \mathrm{d}G_1(t) = H(t). \tag{43}$$

On using (43) in (42) and differentiating the resulting expression with respect to ρ, gives

$$v'(\rho, 0, t) = \frac{4}{\pi}\int_0^t G_1^{-1}(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(\tau)} \frac{\mathrm{d}\eta}{\sqrt{\eta^2-\rho^2}}\int_0^\eta \frac{mS_1(m, t)}{\sqrt{\eta^2-m^2}}\,\mathrm{d}m\,\mathrm{d}\tau \qquad \rho \leqslant a(t). \tag{44}$$

In order to solve (39) we again change ρ to m and multiply both sides by $[m^3/(\rho^2-m^2)^{1/2}]$ and integrate as follows

$$2\int_0^\rho \frac{m^3 S_2(m, t)}{\sqrt{\rho^2-m^2}}\,\mathrm{d}m = \int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi\mathrm{d}\xi\int_0^\rho \frac{m^2 J_1(\xi m)}{\sqrt{\rho^2-m^2}}\,\mathrm{d}m\int_0^{a(\tau)} \lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau \qquad \rho \leqslant a(t). \tag{45}$$

On using a result from Watson[8] and interchanging the order of integration yields

$$2\int_0^\rho \frac{m^3 S_2(m, t)}{}\,\mathrm{d}m = \int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^{a(\tau)} \lambda v(\lambda, 0, \tau)\Big[\sin^{-1}(\rho/\lambda)H(\lambda-\rho) + \frac{\pi}{2}H(\rho-\lambda) - \frac{\rho H(\lambda-\rho)}{\sqrt{\lambda^2-\rho^2}}\Big]\,\mathrm{d}\lambda\,\mathrm{d}\tau. \tag{46}$$

On differentiating (46) with respect to ρ and using the fact that $\rho \leqslant a(t)$, gives

$$2\frac{\partial}{\partial\rho}\int_0^\rho \frac{m^3 S_2(m, t)}{\sqrt{\rho^2-m^2}}\,\mathrm{d}m = -\int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\left\{\rho\frac{\partial}{\partial\rho}\int_\rho^{a(\tau)} \frac{\lambda v(\lambda, 0, \tau)}{\sqrt{\lambda^2-\rho^2}}\,\mathrm{d}\lambda\right\}\mathrm{d}\tau. \tag{47}$$

Again changing ρ to η and multiplying by $[(\eta^2-\rho^2)^{1/2}/\eta]$ and integrating we have

$$2\int_\rho^{a(t)} \frac{\sqrt{\eta^2-\rho^2}}{\eta}\,\mathrm{d}\eta\,\frac{\partial}{\partial\eta}\int_0^\eta \frac{m^3 S_2(m, t)}{\sqrt{\eta^2-m^2}}\,\mathrm{d}m = \frac{\pi}{2}\int_0^t \psi(t-\tau)\int_\rho^{a(\tau)} \lambda v(\lambda, 0, \tau)\,\mathrm{d}\lambda\,\mathrm{d}\tau, \qquad \rho \leqslant a(t). \tag{48}$$

Finally on differentiating (48) with respect to ρ gives

$$v''(\rho, 0, t) = \frac{4}{\pi}\int_0^t \psi^{-1}(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(\tau)} \frac{\mathrm{d}\eta}{\eta\sqrt{\eta^2-\rho^2}}\frac{\partial}{\partial\eta}\int_0^\eta \frac{m^3 S_2(m, \tau)}{\sqrt{\eta^2-m^2}}\,\mathrm{d}m\,\mathrm{d}\tau, \qquad \rho \leqslant a(t), \tag{49}$$

where we have used a similar relation to (43) for $\psi(t)$.

On combining (44) and (49) and further simplifying we get

$$v(\rho, 0, t) = \frac{4}{\pi}\int_0^t [G_1(t-\tau) + \tfrac{1}{2}\psi(t-\tau)]^{-1}\frac{\partial}{\partial\tau}\big[S(\tau)R_e\sqrt{a^2(\tau)-\rho^2}\big]\,\mathrm{d}\tau, \qquad \rho \leqslant a(t). \tag{50}$$

On substituting (50) in (25) and (27), we have

$$u_\theta(\rho, \theta, 0, t) = \frac{4\sin\theta}{\pi}\int_0^t \left[G_1(t-\tau) + \frac{\psi}{2}(t-\tau)\right]^{-1}\frac{\partial}{\partial\tau}\big[S(\tau)R_e\sqrt{a^2(\tau)-\rho^2}\big]\,\mathrm{d}\tau, \qquad \rho \leqslant a(t). \tag{51}$$

In an analogous manner, we find that if, in place of (25) we write the conditions for u_ρ and carry out the similar analysis, we obtain

$$u_\rho(\rho, \theta, 0, t) = -\frac{4\cos\theta}{\pi}\int_0^t \left[G_1(t-\tau) + \frac{\psi}{2}(t-\tau)\right]^{-1} \frac{\partial}{\partial\tau}\{S(\tau)R_e\sqrt{a^2(\tau)-\rho^2}\}\,\mathrm{d}\tau, \qquad \rho \leqslant a(t). \tag{52}$$

Equations (51) and (52) thus give the angular and the radial components of the displacement in the linear theory of viscoelasticity. We note that these reduce to the expressions of the classical elasticity theory[4], when the usual limits of passing from viscoelasticity to classical elasticity are employed.

The relevant stress components in the plane of the crack can now be determined by using (51) and (52). For determining $\sigma_{\theta z}$ we first take the inverse Laplace transform of (33) and obtain

$$\sigma_{\theta z}(\rho, \theta, 0, t) = \left[-\tfrac{1}{2}\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_0(\xi\rho)\int_0^{a(\tau)} \lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau \right.$$
$$\left. -\tfrac{1}{2}\int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \frac{\xi J_1(\xi\rho)}{\rho}\mathrm{d}\xi\int_0^{a(\tau)}\lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau\right]\sin\theta. \tag{53}$$

On substituting the value of $v(\rho, 0, t)$ from (50) in (53) and after considerable manipulations and simplifications we get

$$\sigma_{\theta z}(\rho, \theta, 0, t) = -\frac{2}{\pi}\left[\frac{-a(t)}{\sqrt{\rho^2 - a^2(t)}} + \arcsin\left(a(t)/\rho\right)\right]S(t)\sin\theta$$
$$-\frac{2}{\pi}\left[\int_0^t \frac{\psi(t-\tau)}{2}\frac{\partial}{\partial\tau}\int_0^t\left[G_1(\tau-\eta) + \frac{\psi}{2}(\tau-\eta)\right]^{-1}\mathrm{d}\tau\right.$$
$$\left.\times\frac{\partial}{\partial\eta}\left\{\frac{a^3(\eta)S(\eta)}{\rho^2\sqrt{\rho^2 - a^2(\eta)}}\,\mathrm{d}\eta\right\}\right]\sin\theta \qquad \rho > a(t)$$
$$= -S(t)\sin\theta, \qquad \rho \leqslant a(t). \tag{54}$$

For determining $\sigma_{\rho z}$ we note from (8), (9) and (22) that on $z = 0$

$$\bar{\sigma}_{\rho z}(\rho, \theta, 0, p) = \left[-(1-\bar{v})\frac{\partial^2\bar{\Phi}}{\partial z^2} + \bar{v}\frac{\partial^2\bar{\Phi}}{\partial\rho^2}\right]\cos\theta.$$

On using (32) in this equation and taking the inverse Laplace transform we get

$$\sigma_{\rho z}(\rho, \theta, 0, t) = \left[\frac{1}{2}\int_0^t\left[G_1(t-\tau) + \frac{\psi}{2}(t-\tau)\right]^{-1}\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_0(\xi\rho)\,\mathrm{d}\xi\right.$$
$$\times\int_0^{a(\tau)}\lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau + \frac{1}{4}\int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_0(\xi\rho)\,\mathrm{d}\xi$$
$$\times\int_0^{a(\tau)}\lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau - \frac{1}{2}\int_0^t \psi(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty\frac{\xi J_1(\xi\rho)}{\rho}\mathrm{d}\xi$$
$$\left.\times\int_0^{a(\tau)}\lambda v J_0(\xi\lambda)\,\mathrm{d}\lambda\,\mathrm{d}\tau\right]\cos\theta. \tag{55}$$

Again on using (50) in (55) and after considerable simplifications we find that

$$\sigma_{\rho z}(\rho, \theta, 0, t) = \frac{2}{\pi}\left[\frac{-a(t)}{\sqrt{\rho^2 - a^2(t)}} + \arcsin(a(t)/\rho)\right]S(t)\cos\theta$$

$$+\frac{2}{\pi}\left[\int_0^t \frac{\psi(t-\tau)}{2}\frac{\partial}{\partial\tau}\int_0^\tau\left[G_1(\tau-\eta)+\frac{\psi}{2}(\tau-\eta)\right]^{-1}\frac{\partial}{\partial\eta}\right.$$

$$\left.\times\left\{\frac{S(\eta)a^3(\eta)}{\rho^2\sqrt{\rho^2-a^2(\eta)}}\,\mathrm{d}\eta\right\}\right]\cos\theta, \qquad \rho > a(t)$$

$$= S(t)\cos\theta, \qquad \rho \leqslant a(t). \tag{56}$$

Other stress components can be calculated in a similar manner, but since these would not be of any direct interest at present, we do not record them explicitly here. From equations (54) and (56) we, however, note that these expressions reduce to those obtained in the classical elasticity theory[4] when the usual limiting processes are employed.

The stress intensity factors can now be calculated by using (54) and (56). Thus the stress intensity factor defined by the relation

$$N_1(t) = \lim_{\rho\to a^+(t)} \sqrt{2(\rho - a(t))}\,\sigma_{\theta z}(\rho, \theta, 0, t)$$

is given by

$$N_1(t) = \frac{2}{\pi}\sqrt{a(t)}\,S(t)\sin\theta - \lim_{\rho\to a^+(t)}\sqrt{2(\rho - s(t)}$$

$$\times\left[-\frac{2}{\pi}\int_0^t \frac{\psi(t-\tau)}{2}\frac{\partial}{\partial\tau}\int_0^\tau\left[G_1(\tau-\eta)+\frac{\psi}{2}(\tau-\eta)\right]^{-1}\frac{\partial}{\partial\eta}\right.$$

$$\left.\times\left\{\frac{a^3(\eta)S(\eta)}{\rho^2\sqrt{\rho^2-a^2(\eta)}}\,\mathrm{d}\eta\right\}\mathrm{d}\tau\right]\sin\theta, \qquad \rho > a(t). \tag{57}$$

Also the stress intensity factor defined by

$$N_2(t) = \lim_{\rho\to a^+(t)} \sqrt{2(\rho - a(t))}\,\sigma_{\rho z}(\rho, \theta, 0, t)$$

turns out to be

$$N_2(t) = -\frac{2}{\pi}\sqrt{a(t)}\,S(t)\cos\theta - \lim_{\rho\to a^+(t)}\sqrt{2(\rho - a(t))}$$

$$\times\left[\frac{2}{\pi}\int_0^t \frac{\psi(t-\tau)}{2}\frac{\partial}{\partial\tau}\int_0^\tau\left[G_1(\tau-\eta)+\frac{\psi}{2}(\tau-\eta)\right]^{-1}\frac{\partial}{\partial\eta}\right.$$

$$\left.\times\left\{\frac{a^3(\eta)S(\eta)}{\rho^2\sqrt{\rho^2-a^2(\eta)}}\,\mathrm{d}\eta\right\}\mathrm{d}\tau\right]\cos\theta, \qquad \rho > a(t). \tag{58}$$

If we now define the work done in the opening of the crack to be the work done by external loads acting through the displacements on the boundary of the crack, it follows that

$$W(t) = 2\pi\int_0^{a(t)} \rho S(t)\{-u_\rho\cos\theta + u_\theta\sin\theta\}\,\mathrm{d}\rho. \tag{59}$$

On employing equations (51) and (52) in (59) we obtain

$$W(t) = \frac{8}{3} S^2(t)\, a^3(t) \left[G_1(0) + \frac{1}{2} \psi(0) \right]^{-1}$$
$$+ 8 \int_0^{a(t)} \rho\, S(t) \left\{ \int_0^t S(t-\tau) R_e \sqrt{a^2(t-\tau) - \rho^2}\, \frac{\partial}{\partial \tau} \left[G_1(\tau) + \frac{\psi}{2}(\tau) \right]^{-1} \mathrm{d}\tau \right\} \mathrm{d}\rho. \quad (60)$$

In view of the definitions (9) and (35), if we identify $G_1(0)$ and $\psi(0)$ in the limits to be

$$G_1(0) = 2\mu \quad \text{and} \quad \psi(0) = 2\mu\nu/(1-\nu),$$

it follows from (60) that the first term on the right-hand side of this equation is identical to the one obtained in the classical theory of elasticity[3]. The second term on the right-hand side of (60), therefore, represents the excess amount of work required because of the energy dissipation which takes place in the viscoelastic bodies.

4. FURTHER DISCUSSIONS

In the present section we shall discuss some of the previous results for special kind of viscoelastic materials. The two prototypes selected are the Maxwell model and the standard linear viscoelastic solid.

(a) *Maxwell model*

If we recall equations (9) and (35) we note that $\psi(t)$ involves both $G_1(t)$ and $G_2(t)$. Hence, in order to eliminate $G_2(t)$ in terms of $G_1(t)$ we assume, following Graham[2], that the viscoelastic material has similar behavior in shear and dilation. This assumption implies that the Poisson's ratio ν can now be taken as a constant. Thus from (9) and (35) we have

$$p \frac{\bar{\psi}(p)}{2} = \frac{\nu}{(1-\nu)} p \frac{\bar{G}_1(p)}{2},$$

and hence

$$\psi(t) = \left(\frac{\nu}{1-\nu} \right) G_1(t). \quad (61)$$

Furthermore, if we define $\chi(t) = [G_1(t) + \frac{1}{2}\psi(t)]^{-1}$, it follows that

$$\chi(t) = \frac{2(1-\nu)}{(2-\nu)} G_1^{-1}(t). \quad (62)$$

For a Maxwell material we assume

$$G_1(t) = G_0\, e^{-t/\tau_0}, \qquad G_1^{-1}(t) = \frac{1}{G_0}(1 + t/\tau_0), \quad (63)$$

and, therefore,

$$\psi(t) = \frac{\nu}{(1-\nu)} G_0\, \mathrm{e}^{-t/\tau_0}, \qquad \chi(t) = \frac{2(1-\nu)}{(2-\nu)} \frac{1}{G_0} (1 + t/\tau_0). \quad (64)$$

On employing (64) in (51) and (52) we obtain

$$u_\theta(\rho, \theta, 0, t) = \frac{8(1-\nu)\sin\theta}{\pi(2-\nu)G_0}\{a^2(t)-\rho^2\}^{1/2}S(t) + \frac{8(1-\nu)\sin\theta}{\pi(2-\nu)G_0\tau_0} \times \int_0^t R_e\{a^2(t-\tau)-\rho^2\}^{1/2}S(t-\tau)\,\mathrm{d}\tau, \qquad \rho \leqslant a(t). \tag{65}$$

$$u_\rho(\rho, \theta, 0, t) = -\frac{8(1-\nu)\cos\theta}{\pi(2-\nu)G_0}\{a^2(t)-\rho^2\}^{1/2}S(t) - \frac{8(1-\nu)\cos\theta}{\pi(2-\nu)G_0\tau_0} \times \int_0^t R_e(a^2(t-\tau)-\rho^2)^{1/2}S(t-\tau)\,\mathrm{d}\tau, \qquad \rho \leqslant a(t). \tag{66}$$

Equations (65) and (66) represent the displacement components in the Maxwell material. The first expression on the right-hand side of both the equations corresponds to the elastic solution[4] while the second expression on both the equations represent the viscoelastic effect.

Also, on substituting (64) in (54) we get

$$\sigma_{\theta z}(\rho, \theta, 0, t) = -\frac{2}{\pi}\left[\frac{-a(t)}{\sqrt{\rho^2-a^2(t)}} + \arcsin(a(t)/\rho)\right]S(t)\sin\theta - \left\{\frac{2}{\pi}\frac{2(1-\nu)}{(2-\nu)}\frac{\nu}{(1-\nu)}\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\tau G_1^{-1}(\tau-\eta)\frac{\partial}{\partial\eta}\left[\frac{a^3(\eta)S(\eta)\,\mathrm{d}\eta}{2\rho^2\sqrt{\rho^2-a^2(\eta)}}\right]\mathrm{d}\tau\right\} \times \sin\theta, \qquad \rho > a(t),$$

which on further simplification reduces to

$$\sigma_{\theta z}(\rho, \theta, 0, t) = -\frac{2}{\pi}\left[\frac{-a(t)}{\sqrt{\rho^2-a^2(t)}} + \arcsin(a(t)/\rho) + \frac{\nu}{(2-\nu)}\frac{a^3(t)}{\rho^2\sqrt{\rho^2-a^2(t)}}\right] \times S(t)\sin\theta, \qquad \rho > a(t). \tag{67}$$

In a similar manner (56) becomes

$$\sigma_{\rho z}(\rho, \theta, 0, t) = \frac{2}{\pi}\left[\frac{-a(t)}{\sqrt{\rho^2-a^2(t)}} + \arcsin(a/(t)\rho) - \frac{\nu}{(2-\nu)}\frac{a^3(t)}{\rho^2\sqrt{\rho^2-a^2(t)}}\right]S(t)\cos\theta, \qquad \rho > a(t). \tag{68}$$

The stress intensity factors (57) and (58) can now be handled in the complete form, and these turn out to be

$$N_1(t) = \frac{4}{\pi}\frac{(1-\nu)}{(2-\nu)}\sqrt{a(t)}\,S(t)\sin\theta, \tag{69}$$

$$N_2(t) = -\frac{4}{\pi}\frac{\sqrt{a(t)}}{(2-\nu)}S(t)\cos\theta. \tag{70}$$

Finally, the expression (60) for the work done in opening the crack reduces to

$$W(t) = \frac{16(1-\nu)}{3(2-\nu)G_0}S^2(t)a^3(t) + \frac{16(1-\nu)}{3(2-\nu)G_0\tau_0}\int_0^{a(t)}\rho\,\mathrm{d}\rho\int_0^t R_e\sqrt{a^2(t-\tau)-\rho^2}\,S(t-\tau)\,\mathrm{d}\tau. \tag{71}$$

(b) *Standard linear solid*

For a standard linear solid we assume the constitutive equations in shear to be of the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \frac{1+f}{\tau_0}\sigma = 2\mu\left(\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{e}{\tau_0}\right), \tag{72}$$

where τ_0, μ and f are all constants. Corresponding to this equation we find

$$G_1(t) = \frac{2\mu}{(1+f)}\left[1 + f\exp\left\{-\frac{1+f}{\tau_0}t\right\}\right], \tag{73}$$

$$G^{-1}(t) = \frac{(1+f)}{2\mu}\left[1 - \frac{f}{1+f}\exp\left(-t/\tau_0\right)\right]. \tag{74}$$

Equations (61) and (62), in the present case, take the form

$$\psi(t) = \frac{2\mu\nu}{(1-\nu)(1+f)}\left[1 + f\exp\left\{-\frac{1+f}{\tau_0}t\right\}\right]$$

$$\chi(t) = \frac{(1-\nu)(1+f)}{\mu(2-\nu)}\left[1 - \left(\frac{f}{1+f}\right)\exp(-t/\tau_0)\right]. \tag{75}$$

On employing (75) in (51) and (52) we get

$$u_\theta(\rho, \theta, 0, t) = \frac{4(1-\nu)}{\pi(2-\nu)\mu}\sqrt{a^2(t) - \rho^2}\, S(t)\sin\theta + \left[\frac{4}{\pi}\frac{(1-\nu)f}{(2-\nu)\mu\tau_0}\int_0^t R_e\{a^2(t-\tau) - \rho^2\}^{1/2} S(t-\tau)\exp(-\tau/\tau_0)\,\mathrm{d}\tau\right]\sin\theta$$
$$\rho \leqslant a(t). \tag{76}$$

$$u_\rho(\rho, \theta, 0, t) = -\frac{4}{\pi}\frac{(1-\nu)}{(2-\nu)\mu}\sqrt{a^2(t) - \rho^2}\, S(t)\cos\theta - \left[\frac{4}{\pi}\frac{(1-\nu)f}{(2-\nu)\mu\tau_0}\int_0^t R_e\{a^2(t-\tau) - \rho^2\}^{1/2} S(t-\tau)\exp(-\tau/\tau_0)\,\mathrm{d}\tau\right]\cos\theta,$$
$$\rho \leqslant a(t). \tag{77}$$

In the present case also the last expressions on the right hand sides of (76) and (77) show the effect of viscoelasticity. The expressions for the stress components and the stress intensity factors remain the same. The expression for $W(t)$, however, reduces to

$$W(t) = \frac{8(1-\nu)}{3(2-\nu)\mu}S^2(t)a^3(t) + \frac{8(1-\nu)f}{(2-\nu)\mu\tau_0}\int_0^{a(t)}\rho\,\mathrm{d}\rho\int_0^t R_e\sqrt{a^2(t-\tau) - \rho^2} \times S(t-\tau)\exp(-\tau/\tau_0)\,\mathrm{d}\tau. \tag{78}$$

5. CONCLUDING REMARKS

We point out the the analysis presented in the paper is valid only under the assumption that the crack propagates in a self-similar fashion in the same plane. Based on the energetic arguments and assuming that the crack propagates in the coplanar manner, Smith[9] has concluded that for linear elastic materials the most favorable growth mode is that for which the circular periphery becomes an ellipse, such that there is no growth perpendicular to the direction of the application of the shear stress. Smith's analysis, however, shows that for small values of the Poisson's ratio, ν, the difference between assuming a uniform growth with the crack retaining a circular periphery and the growth with the crack changing to elliptic periphery is not considerably significant. On the other hand the work of Erdogan and Sih[10], for a line crack in a flat plate under pure shear, indicates that the crack does not remain coplanar but extends in a curved fashion. This point has also been raised by Sih and Liebowitz[11] who, while commenting on the work of Segedin[3], remarked that the assumption that the shape of circular crack remains circular is not versatile enough. In analysing the values of the critical stresses to establish the conditions of fracture in an elliptic crack, these authors, however, also assumed that elliptic crack propagates in a coplanar manner into another ellipse having the same foci as that of the original elliptic crack,

In view of the above remarks it follows that the expression for the work done in opening the crack, which can be related to the fracture criterion in the energetic approach, will get modified if the effect of curvedness, etc. are also taken into consideration. In fact, as appears from the work of Sih and Leibowitz[11], indications are that a criterion of fracture, for the skew-symmetric problems, will have to depend upon certain combinations of the stress intensity factors $N_1(t)$ and $N_2(t)$. Clearly, such information has to come from experimental observations.

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Абстракт — В работе исследуется задача линейного вязкоупругого тела, в котором находится дискообразная трещина, подверженная действию направленного сдвига к краю трещины. Определяется выражения в замкнутом виде для перемещения выше прверхности трещины, компоненты сдвига в плоскости трещины и факторы интенсивности напряжений. Далее, приспосаблинается разные вйражения для двух частных линейный вязкоупругих материалов и, куда это возможно, указывается эффект вязкоупругости.